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On a class of exact two-dimensional stationary solutions for the Broadwell model of the Boltzmann equation

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Abstract. Potential flows in steady-state conditions are studied in detail in the framework of the plane Broadwell discrete-velocity model of the Boltzmann equation. It is shown that explicit exact solutions with physical meaning can be obtained in terms of elementary or special functions. Previously discovered analytical incompressible flows are recovered as particular cases. New solutions for compressible flows are classified and discussed.

1. Introduction

The first discrete-velocity models of the Boltzmann equation were proposed by Carleman [1] and Broadwell [2]. This approach to problems of kinetic theory was essentially developed and generalized in the 1970s [3, 4] and has become very popular in the last ten years. In spite of their visible simplicity, discrete-velocity models are sufficiently complicated from a mathematical point of view (see [5] for a review). There are many open problems, even for the quite simple and popular Broadwell model, and some of these problems can be clarified on the basis of exact solutions. The present paper is devoted to the construction and investigation of a new class of exact solutions of the 2D stationary Broadwell model. The first solutions for this model were derived in [6] as particular cases of time-dependent solutions. It should be noted that such solutions, together with several exact solutions determined by Cornille for other discrete-velocity models [7], and other earlier discovered solutions to the Carleman model [8, 9], definitely have common mathematical structure. In fact, the majority of the known exact solutions for discrete-velocity models belong to the same class. This was discussed in detail in [10], where other possibilities for constructing exact solutions were also indicated, in particular stationary potential flows for the Broadwell model. In the present paper, we realize that idea and construct a wide class of new exact solutions. It is shown that some of our solutions correspond to incompressible flows which have been investigated in detail in [10, 11]. On the other hand, we provide below more complex classes of exact solutions corresponding to compressible flows of the Broadwell gas. The structure of these new solutions and their properties are quite different from those of the other already known exact solutions.

2. Stationary potential flows for the plane Broadwell model

We consider the four-velocity plane Broadwell-model equations in dimensionless form with velocities rotated by $(2j-1)\pi/4$, $j = 1, 2, 3, 4$, with respect to the x axis, and common unit speed for the densities $f_j(x, y)$ in steady-state conditions

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y} = -\frac{\partial f_3}{\partial x} - \frac{\partial f_3}{\partial y} = \frac{\partial f_2}{\partial x} - \frac{\partial f_2}{\partial y} = -\frac{\partial f_4}{\partial x} + \frac{\partial f_4}{\partial y} = \sqrt{2}(f_2 f_4 - f_1 f_3). \tag{2.1}$$

For the new dependent variables

$$\rho = f_1 + f_2 + f_3 + f_4 \qquad p = f_1 - f_2 + f_3 - f_4 \tag{2.2}$$

$$u = \frac{1}{\sqrt{2}}(f_1 - f_2 - f_3 + f_4) \qquad v = \frac{1}{\sqrt{2}}(f_1 + f_2 - f_3 - f_4) \tag{2.3}$$

where ρ is the total density and (u, v) is the mean velocity vector, the governing equations decouple as

$$\rho_x + p_y = 0 \qquad p_x + \rho_y = 0 \qquad u_x + v_y = 0 \qquad v_x + u_y = 2uv - p\rho. \tag{2.4}$$

The general solution of the first two equations has the form

$$\rho(x, y) = A(x + y) + B(x - y) \qquad p(x, y) = -A(x + y) + B(x - y) \tag{2.5}$$

where $A = f_2 + f_4$ and $B = f_1 + f_3$ are arbitrary smooth non-negative functions. Then

$$p\rho = -A^2(x + y) + B^2(x - y) = -2\theta(x, y) \qquad \theta_{xx} = \theta_{yy}. \tag{2.6}$$

We now introduce the stream function $\Psi(x, y)$ by [12]

$$u = \Psi_y(x, y) \qquad v = -\Psi_x(x, y) \tag{2.7}$$

and restrict ourselves to the so-called potential flows. Let $\Phi(x, y)$ be a potential function for our system, i.e. Φ satisfies the Laplace equation $\Phi_{xx} + \Phi_{yy} = 0$ and

$$u = \Phi_x(x, y) \qquad v = \Phi_y(x, y). \tag{2.8}$$

The complex function of a complex variable

$$f(z) = \Phi(x, y) + \mathrm{i}\Psi(x, y) \qquad z = x + \mathrm{i}y \tag{2.9}$$

is analytic, since Φ and Ψ are conjugate harmonic functions, and the velocity components are defined by $u - \mathrm{i}v = f'(z)$, or

$$u = \mathrm{Re}\, f'(z) \qquad v = -\mathrm{Im}\, f'(z). \tag{2.10}$$

The last equation in (2.4) can now be written as

$$\Phi_{xy} = \Phi_x\Phi_y + \theta \tag{2.11}$$

which, since $[f'(z)]^2 = \Phi_x^2 - \Phi_y^2 - \mathrm{i}\Phi_x\Phi_y$ and $f''(z) = \Phi_{xx} - \mathrm{i}\Phi_{xy}$, takes the form

$$\mathrm{Im}[f''(z) - \tfrac{1}{2}(f'(z))^2] = -\theta(x, y) \tag{2.12}$$

so that the function $\theta(x, y)$ also satisfies the Laplace equation as the imaginary part of an analytic function. This implies that the equations $\theta_{xx} = \theta_{yy} = 0$ are easily integrated to yield

$$\theta(x, y) = \mathrm{Im}(\alpha z^2 + \beta z + \gamma) \tag{2.13}$$

where α, β, γ are arbitrary (complex) constants and $\mathrm{Im}\,\alpha = 0$.

Setting $\beta = \beta_1 + \mathrm{i}\beta_2$, $\gamma = \gamma_1 + \gamma_2$, and bearing equation (2.6) in mind, the functions A and B are determined as

$$A^2(x) = \alpha x^2 + (\beta_1 + \beta_2)x + \gamma_2 + N \qquad B^2(x) = \alpha x^2 + (\beta_1 - \beta_2)x - \gamma_2 + N \tag{2.14}$$

where N is an additional real integration constant.

In conclusion, by analytic continuation, the most general class of potential flows for the stationary Broadwell-model equations (2.1) are described by the nonlinear first-order equation in the complex plane

$$f''(z) - \tfrac{1}{2}[f'(z)]^2 + (\alpha z^2 + \beta z + \gamma) = 0 \tag{2.15}$$

which can be reduced to the linear equation

$$F''(z) - \tfrac{1}{2}(\alpha z^2 + \beta z + \gamma)F(z) = 0 \tag{2.16}$$

by the substitution

$$f(z) = -2\ln F(z). \tag{2.17}$$

The general solution of equation (2.16) can be expressed in terms of elementary or special functions [13], so that a wide class of analytical solutions can be obtained. The distribution function follows as

$$\begin{aligned}
f_1(x, y) &= \tfrac{1}{2}B(x - y) + \frac{u(x, y) + v(x, y)}{2\sqrt{2}} \\
f_2(x, y) &= \tfrac{1}{2}A(x + y) - \frac{u(x, y) - v(x, y)}{2\sqrt{2}} \\
f_3(x, y) &= \tfrac{1}{2}B(x - y) - \frac{u(x, y) + v(x, y)}{2\sqrt{2}} \\
f_4(x, y) &= \tfrac{1}{2}A(x + y) + \frac{u(x, y) - v(x, y)}{2\sqrt{2}}.
\end{aligned} \tag{2.18}$$

3. Positivity conditions

It is clear that only non-negative densities $f_j(x, y)$, $j = 1, 2, 3, 4$, have a physical meaning. On the other hand, our solutions depend on the six real parameters in α, β, γ, N and on two complex integration constants from the solution of equation (2.16) for F. It is shown here that, for any given domain D in the complex plane where the function f is analytic, the previous free parameters can be chosen in such a way that positivity is fulfilled everywhere in D.

First of all, the functions A^2 and B^2 in (2.14) must be non-negative. This can be achieved by taking N positive and large enough if D is bounded. The latter restriction can even be relaxed when $\alpha > 0$, or when $\alpha = \beta = 0$. Moreover, due to equation (2.18), the *functions $u(x, y)$ and $v(x, y)$ have to satisfy the following conditions*

$$(u+v)^2 \leqslant 2B^2(x-y) \qquad (u-v)^2 \leqslant 2A^2(x+y). \tag{3.1}$$

Let $f(z)$ be a solution of (2.15). There then exists a positive constant M such that

$$|f'(z)|^2 \leqslant M \qquad \forall z \in D. \tag{3.2}$$

By the same positivity argument as before, and with the same possible restrictions, we can always choose the number N sufficiently large to satisfy the stronger conditions

$$B^2(x-y) \geqslant M \qquad A^2(x+y) \geqslant M \qquad x+\mathrm{i}y \in D. \tag{3.3}$$

Then, the positivity conditions (3.1) will also be satisfied for any $z \in D$ because of the standard estimate

$$(u \pm v)^2 \leqslant 2(u^2+v^2) = 2|f'(z)|^2 \leqslant 2M. \tag{3.4}$$

Thus, we obtain positive solutions to the Broadwell model for any given solution $f(z)$ of equation (2.15). Condition (3.2) for the analytic function $f(z)$ is not valid near singular points (including $z = \infty$), so that these points should be considered separately for each specific function $f(z)$. It is important to stress, however, that for any bounded open domain D and for any given $\epsilon > 0$, we can choose M in (3.2) sufficiently large to satisfy equation (3.2) itself for all $z \in D$ with the exception of a finite number p of small neighbourhoods $|z - z_k| < \epsilon$ of the singular points $z_k \in \overline{D}$, $k = 1, 2, \ldots, p$ of the function $f(z)$.

4. Exact solutions and their classification

We consider equation (2.16) in three different cases: $\alpha = \beta = 0$, $\gamma \neq 0$; $\alpha = \gamma = 0$, $\beta \neq 0$; and $\beta = 0$, $\alpha \neq 0$. It is clear that the general case can be reduced to one of these cases by a substitution $z = \tilde{z} +$ constant.

The simplest case is $\alpha = \beta = 0$, $\gamma \neq 0$, and corresponds to incompressible flows ($\rho =$ constant) of the Broadwell gas. We put $\gamma = 2\delta^2$ and obtain the general solution of (2.16), $f(z) = -2\ln[\sinh\delta(z+z_0)] + \text{constant}$, $f'(z) = -2\delta\coth\delta(z+z_0)$. It is easily seen that, by a rigid translation and rotation of the (x, y) plane, we can always refer to the case when δ is real and positive and $z_0 = 0$, and that, by a further coordinate scaling, we may even take $\delta = 1$. In fact, upon the substitution

$$\tilde{u} + \mathrm{i}\tilde{v} = \frac{\delta}{|\delta|^2}(u + \mathrm{i}v) \qquad \tilde{z} = \delta(z + z_0) \tag{4.1}$$

we obtain

$$\tilde{u} - \mathrm{i}\tilde{v} = -\frac{\sinh\tilde{x} - \mathrm{i}\sin\tilde{y}}{\cosh^2\tilde{x} - \cos^2\tilde{y}} \tag{4.2}$$

which is a well known solution [5, 11] that corresponds to earlier solutions [8, 9] for the Carleman model. It is clear that even the trivial solution $f = -2\ln z + \text{constant}$, relevant to the limiting case $\gamma = 0$, yields the non-trivial flow described by

$$u - \mathrm{i}v = -\frac{2}{z} = -2\frac{x - \mathrm{i}y}{x^2 + y^2}. \tag{4.3}$$

The solutions (4.2), (4.3) and related boundary-value problems were investigated in detail in the literature, and, therefore, we do not discuss them here. We note only that (4.2), restricted to $\tilde{x} > 0$ and $|\tilde{y}| < \pi$, corresponds to gas outflow from a half-infinite channel through an infinitely small hole in its end-wall, with singular point (sink) at $(0, 0)$ and stagnation points at $(0, \pm\pi)$. The solution (4.3), restricted to $x > 0$, describes the limiting case of gas outflow from a half-space, with sink at $(0, 0)$. It is worth remarking that the most general incompressible potential flow for the Broadwell model is amenable, by translations and rotations, to the above physical situations.

Let us now pass to the cases when α or β is different from zero, leading thus to compressible flows, for which, to our knowledge, exact solutions of the Broadwell model do not exist in the literature. For $\alpha = \gamma = 0$, $\beta \neq 0$, equation (2.16) can be cast, by standard transformations, as a Bessel equation, and, with $\beta = 2\delta^2$, the general solution has the form

$$F(z) = \sqrt{z}[C_1 I_{1/3}(\tfrac{2}{3}\delta z^{3/2}) + C_2 I_{-1/3}(\tfrac{2}{3}\delta z^{3/2})] \tag{4.4}$$

where $I_{\pm 1/3}$ is the well known modified Bessel function of the first kind [13]. We can also use the equivalent representation in terms of Airy functions. In spite of fractional powers, F turns out to be an entire function of z. By a suitable rotation and rescaling of the (x, y) plane, we can always reduce the analysis to the case $\delta = 1$ (new variables will be labelled with the old symbols, for convenience). For brevity, we consider here only the case $C_2 = 0$. Similar conclusions hold when $C_1 = 0$ (notice that the results are independent of C_1, or of C_2, respectively). Then, apart from a non-essential multiplicative constant, $F(z)$ is given by [13]

$$z\prod_{m=1}^{\infty}\left(1 - \frac{z^3}{\lambda_m^3}\right) \tag{4.5}$$

where $\lambda_m^3 = -(3z_m^{(1/3)}/2)^2$ and $z_m^{(1/3)}$, $m = 1, 2, \ldots$, is the countable sequence of the zeros of the function $z^{-1/3}J_{1/3}(z)$. Thus, the solution can be written as

$$u - \mathrm{i}v = -\frac{2}{z} - 6z^2\sum_{m=1}^{\infty}\frac{1}{z^3 - \lambda_m^3} \tag{4.6}$$

so that we obtain, in this case, a complex flow with an infinite number (in the whole plane) of poles $z_m^0 = \lambda_m$, $z_m^{\pm} = \lambda_m \exp(\pm 2\pi\mathrm{i}/3)$, $m = 1, 2, \ldots$, in addition to $z = 0$. The behaviour near each pole can be described by standard methods [12]. Regarding the density, it is given by

$$\rho(x, y) = \sqrt{N + 2(x + y)} + \sqrt{N + 2(x - y)} \tag{4.7}$$

and is positive in the quadrant $-(N/2) - x < y < (N/2) + x$, increasing when x tends to $+\infty$.

In the case $\beta = 0$, $\alpha \neq 0$, the general solution of equation (2.16) can be written in terms of the confluent hypergeometric function ${}_1F_1$. More precisely

$$\exp(\sqrt{\alpha}z^2/2\sqrt{2})F(z)$$

is given by

$$C'{}_1F_1\left(\frac{1+\frac{\gamma}{\sqrt{2\alpha}}}{4}, \frac{1}{2}; \sqrt{\frac{\alpha}{2}}z^2\right) + C''z\,{}_1F_1\left(\frac{3+\frac{\gamma}{\sqrt{2\alpha}}}{4}, \frac{3}{2}; \sqrt{\frac{\alpha}{2}}z^2\right). \tag{4.8}$$

We may take $\alpha = 2$, put $\gamma = 2\lambda$ and consider cases with either $C' = 0$ or $C'' = 0$ for which the analytical solution becomes explicit. Note that the particle number density is given by (2.5) and (2.14), i.e.

$$\rho(x, y) = \sqrt{2(x-y)^2 + N - 2\,\mathrm{Im}\,\lambda} + \sqrt{2(x+y)^2 + N + 2\,\mathrm{Im}\,\lambda} \tag{4.9}$$

where N is an arbitrarily large positive number, increasing when x or y increase. We describe here some simple cases, resorting to the well known formulae [13] (multiplicative constants have no influence on u and v)

$${}_1F_1\left(-n, \tfrac{1}{2}; x^2\right) \propto H_{2n}(x) \qquad x\,{}_1F_1\left(-n, \tfrac{3}{2}; x^2\right) \propto H_{2n+1}(x) \tag{4.10}$$

$n = 0, 1, \ldots$, $H_m(x)$ being the classical mth Hermite polynomial. The two linearly-independent solutions in (4.8) are just

$$\exp\left(-\frac{z^2}{2}\right){}_1F_1\left(\frac{1+\lambda}{4}, \frac{1}{2}; z^2\right) \qquad \exp\left(-\frac{z^2}{2}\right)z\,{}_1F_1\left(\frac{3+\lambda}{4}, \frac{3}{2}; z^2\right). \tag{4.11}$$

It is sufficient to consider the case $\mathrm{Re}\,\lambda \leqslant 0$, because the substitution $\tilde{\lambda} = -\lambda$, $\tilde{z} = \mathrm{i}z$ does not change equation (2.16) when $\beta = 0$. There is, thus, the following countable sequence of solutions involving polynomials

$$\lambda = -(2n+1) \qquad F_n(z) = H_n(z)\exp(-z^2/2) \qquad n = 0, 1, \ldots \tag{4.12}$$

for which, in addition, $\mathrm{Im}\,\lambda = 0$ (see equation (9)). The velocity-field components are defined by (2.10), which yields, in the polynomial case,

$$u - \mathrm{i}v = 2\left[z - 2n\frac{H_{n-1}(z)}{H_n(z)}\right] \qquad n = 0, 1, \ldots. \tag{4.13}$$

In particular, the first few are

$$\begin{aligned} u - \mathrm{i}v &= 2z \\ u - \mathrm{i}v &= 2\left(z - \frac{1}{z}\right) \\ u - \mathrm{i}v &= 2\left(z - \frac{4z}{2z^2-1}\right) \\ u - \mathrm{i}v &= 2\left[z - \frac{3(2z^2-1)}{z(2z^2-3)}\right] \end{aligned} \tag{4.14}$$

for $n = 0, 1, 2, 3$, respectively. Notice that for $n = 0$, this is the typical textbook example of potential flow, where streamlines are hyperbolae and the origin is the stagnation point. In the other examples, this field is superimposed on increasingly complicated flows, exhibiting a finite number (equal to the index n) of simple poles located at the zeros of the Hermite polynomial H_n (which are real). In the global field, there are $n+1$ stagnation points corresponding to the zeros of $zH_n(z) - 2nH_{n-1}(z)$. The first superimposed singular field just coincides with that already obtained in equation (4.3) for different values of the parameters.

Finally, we can describe the elementary solutions (4.13) in terms of the usual Broadwell model with velocities rotated by $j\pi/2$, $j = 1, 2, 3, 4$, with respect to the x axis. To this end, we transform equation (2.1) into new variables

$$x' = \frac{x+y}{\sqrt{2}} \qquad y' = \frac{-x+y}{\sqrt{2}} \tag{4.15}$$

and obtain the standard stationary Broadwell equations

$$\frac{\partial f_1}{\partial x} = -\frac{\partial f_3}{\partial x} = -\frac{\partial f_2}{\partial y} = \frac{\partial f_4}{\partial y} = f_2 f_4 - f_1 f_3 \tag{4.16}$$

where the prime has been omitted. In the new setting, the exact solutions (4.13) read as

$$\begin{aligned} f_1(x, y) &= \sqrt{N + y^2} - y + \operatorname{Im}\psi_n(z) \\ f_2(x, y) &= \sqrt{N + x^2} - x + \operatorname{Re}\psi_n(z) \\ f_3(x, y) &= \sqrt{N + y^2} + y - \operatorname{Im}\psi_n(z) \\ f_4(x, y) &= \sqrt{N + x^2} + x - \operatorname{Re}\psi_n(z) \end{aligned} \tag{4.17}$$

where $z = x + \mathrm{i}y$ and

$$\psi_n(z) = \exp\left(-\frac{\mathrm{i}\pi}{4}\right)\phi_n\left[z\exp\left(\frac{\mathrm{i}\pi}{4}\right)\right] \qquad \phi_n(z) = 2n\frac{H_{n-1}(z)}{H_n(z)} \tag{4.18}$$

for $n = 0, 1, \ldots$. It is remarkable that, for $n = 0$, we obtain a local equilibrium solution, which satisfies identically $f_1 f_3 = f_2 f_4$. We also note that

$$\psi_n(z) = -\mathrm{i}\sum_{k=1}^{n} \frac{1}{z - w_k^{(n)}\exp(-\mathrm{i}\pi/4)} \tag{4.19}$$

where $w_k^{(n)}$ is the kth zero ($1 \leqslant k \leqslant n$) of the Hermite polynomial $H_n(x)$. The n poles of the exact solution (4.17) are then localized at the points $w_k^{(n)}\exp(-\mathrm{i}\pi/4)$. In particular, there is one pole $z = 0$ for $n = 1$, there are two at $\pm(1-\mathrm{i})/2$ for $n = 2$, and so on. In general, the solution can be written as a sum of the local equilibrium solution and n terms which each correspond to a simple pole. Qualitatively, these solutions are similar (they differ only in the number of poles) to the simplest case $n = 1$, which is explicitly written below

$$\begin{aligned} f_1(x, y) &= \sqrt{N + y^2} - y - \frac{x}{x^2 + y^2} \\ f_2(x, y) &= \sqrt{N + x^2} - x - \frac{y}{x^2 + y^2} \\ f_3(x, y) &= \sqrt{N + y^2} + y + \frac{x}{x^2 + y^2} \\ f_4(x, y) &= \sqrt{N + x^2} + x + \frac{y}{x^2 + y^2}. \end{aligned} \tag{4.20}$$

For any given $\epsilon > 0$, we can always choose $N = N(\epsilon)$, such that $f_j > 0$, $(j = 1, 2, 3, 4)$ if $(x^2 + y^2) > \epsilon^2$. From a physical point of view, equation (4.20) describes the outflow through an infinitely small sink in the origin, so that the restriction $(x^2 + y^2) > \epsilon^2$ amounts to considering the finite size of the hole.

5. Conclusions

Thus, the complete description of stationary potential flows for the plane Broadwell model is given. These flows can be separated into two classes: incompressible and compressible. The first class corresponds to elementary transformations of already known exact solutions, while the solutions of the second class described above are essentially new. The most interesting and relatively simple solutions of this class are expressed explicitly in terms of elementary or special functions, and a countable set of rational solutions are described in section 4. We also proved that all our solutions have a physical meaning, i.e. they correspond to non-negative particle densities for the Broadwell model on the desired domain. A more detailed analysis of the physical applications and implications of this model and its exact solutions will be the subject of a future paper, along the lines of which discrete-velocity models have already been applied to fluid-dynamical problems in the literature (see, for instance, [14]). It is worth mentioning here, however, that the reduced dimensionality of our model is not an actual restriction, since it is easy to check that the more realistic three-dimensional Broadwell-model equations for stationary-plane flows can be reduced in the general case to the same equations (4.17).

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